# Mathematics for Engineers II. lectures

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Laplace transform

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# Laplace transform

Laplace transform is a possible tool for solving linear ODE with constant coefficients.The process consists three main steps:

1. The given "hard" problem is transformed into a "simple" equation.
2. This simple equation is solved by purely algebraic manipulations.
3. The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem.

# Laplace transform

## Definition

Let $f(t)$ be defined for $t \geq 0$ ($f(t)$ can be both real or complex). Then the **Laplace transform of** $f$ is defined by the following equation (if the integral is convergent)

$$\mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt.$$

If $F$ is the Laplace transform of $f$, then $f$ is called the **inverse Laplace transform of** $F$ and write $\mathcal{L}^{-1}[F(s)]$.

In practice, we use Laplace transformation tables for the calculation of inverse transform.

# Laplace transform, Example, Exercises

Let $f(t) = 1$. Then

$$\mathcal{L}[f] = \int_0^\infty 1 \cdot e^{-st} dt = \left[ -\frac{1}{s} e^{-st} \right]_{t=0}^\infty = \lim_{t\to\infty} \frac{-1}{s} e^{-st} + \frac{1}{s} = \frac{1}{s}.$$

Obtain the inverse Laplace transform of the following functions!

1. $f(t) = e^{at}$.
2. $f(t) = t$.
3. $f(t) = \cos(at)$.
4. $f(t) = e^{t^2}$.

Obtain the Laplace transform of the following functions using the previous exercises and/or Laplace transformation tables!

1. $F(s) = \frac{1}{s-1}$.
2. $F(s) = \frac{6}{s^4}$.
3. $F(s) = \frac{s}{s^2+4}$.

# Laplace transform, Properties

## Theorem

Both the Laplace transformation and the inverse Laplace transformation are linear.

## Exercises

Obtain the Laplace transform of the following functions!

1. $f(t) = 3t + e^t$.
2. $f(t) = -2 + cost$.

Obtain the inverse Laplace transform of the following functions!

1. $F(t) = \frac{3}{s-2} + \frac{2}{s^2}$.
2. $F(s) = -\frac{12}{s^4}$.

# Laplace transform, Properties

## Shifting

$$\mathcal{L}[e^{\alpha t} f(t)](s) = \mathcal{L}[f(t)](s-\alpha), \qquad s > a + \alpha.$$

## Laplace transform of the derivative

$$\mathcal{L}[f'(t)](s) = s\mathcal{L}[f(t)](s) - f(0).$$

$$\mathcal{L}[f''(t)](s) = s^2\mathcal{L}[f(t)](s) - sf(0) - f'(0).$$

$$\mathcal{L}[f^{(n)}(t)](s) = s^n\mathcal{L}[f(t)](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

# Laplace transform, Partial fractions

Let $N(s)$ and $D(s)$ are polynomials. Our goal is to write $R(s) = \frac{N(s)}{D(s)}$ into a sum of simpler expressions whose inverse Laplace transform can be recognized from a table of Laplace transformation pairs. This simpler fractions are called **partial fractions**. We can assume without losses that the leading coefficient of $D(s)$ and $N(s)$ is 1.

## Steps of the method

Step 1 Find polynomials $r(s)$ and $q(s)$ such that

$$R(s) = \frac{N(s)}{D(s)} = q(s) + \frac{r(s)}{D(s)},$$

where the degree of $r(s)$ is strictly less than the degree of $D(s)$.

# Laplace transform, Partial fractions

## Steps of the method

Step 2 Write $D(s)$ as a product of factors $(s-b)^n$ or $(s^2+\alpha s+\beta)$ where $\alpha,\ \beta$ and $b$ are real numbers and $(s^2+\alpha s+\beta)$ has no real zeros.

Step 3 Decompose $\frac{r(s)}{D(s)}$ into a sum of partial fractions in the following way:

(i) For each factor of the form $(s-b)^n$ write

$$\frac{A_1}{(s-b)}+\frac{A_2}{(s-b)^2}+\cdots+\frac{A_n}{(s-b)^n}.$$

(ii) For each factor of the form $(s^2+\alpha s+\beta)$ write

$$\frac{B_1 s+C_1}{s^2+\alpha s+\beta}+\frac{B_2 s+C_2}{(s^2+\alpha s+\beta)^2}+\cdots+\frac{B_n s+C_n}{(s^2+\alpha s+\beta)^n}.$$

# Laplace transform, Partial fractions

## Exercises

Decompose into partial fractions $R(s)$, where

1. $R(s) = \frac{s^3+s^2+2}{s^2-1}$;
2. $R(s) = \frac{s^2+5s-3}{(s^2+16)(s-2)}$.

## Exercises

1. Find $\mathcal{L}^{-1}\left[\frac{1}{s(s-3)}\right]$.
2. Find $\mathcal{L}^{-1}\left[\frac{3s+6}{s^2+3s}\right]$.

# Laplace transform, solution of ODE

## Example

Use Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = e^{-t}, \quad y(0) = y'(0) = 0.$$

**Solution:** By the linearity of the Laplace transform we can write:

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[e^{-t}].$$

Using the initial data and the identities for the Laplace transformation of the derivative we have:

$$s^2 Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s+1}, \quad \text{where} \quad \mathcal{L}[y] = Y(s).$$

# Laplace transform, solution of ODE

Rearranging gives

$$Y(s) = \frac{1}{(s+1)(s^2+3s+2)}$$

and

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+3s+2)}\right].$$

Using the method of partial fractions we can write

$$\frac{1}{(s+1)(s^2+3s+2)} = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}.$$

Thus,

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-2t} - e^{-t} + te^{-t}.$$